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# Lax pair representation and Darboux transformation of noncommutative Painlevé second equation

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## Abstract

The extension of Painlevé equations to noncommutative spaces has been considering extensively in the theory of integrable systems and it is also interesting to explore some remarkable aspects of these equations such as Painlevé property, Lax representation, Darboux transformation and their connection to well know integrable equations. This paper is devoted to the Lax formulation, Darboux transformation and Quasideterminant solution of noncommutative Painlevé second equation which is recently introduced by V. Retakh and V. Rubtsov.

*Keywords:*

Integrable systems, Lax equation, Darboux transformation, quasideterminants and solitons .

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## 1. Introduction

The Painlevé equations were discovered by Painlevé and his colleagues when they were classifying the nonlinear second-order ordinary differential equations with respect to their solutions [1]. The importance of Painlevé equations from mathematical point of view is because of their frequent appearance in the various areas of physical sciences including plasma physics, fiber optics, quantum gravity and field theory, statistical mechanics, general relativity and nonlinear optics. The classical Painlevé equations are regarded as completely integrable equations and obeyed the Painlevé test [2,3,4]. These equations are subjected to the some properties such as linear representation, hierarchy, Darboux transformation(DT) and Hamiltonian structure because of their reduction from integrable systems, i.e, Painlevé second (P-II) equation arises as reduction of KdV equation [5,6].

The Noncommutative(NC) extension of Painlevé equations is quite interesting in order to explore their properties which they possess on ordinary spaces. NC spaces are characterized by the noncommutativity of the spatial co-ordinates, if  $x^\mu$  are the space co-ordinates then the noncommutativity is defined by  $[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$  where parameter  $\theta^{\mu\nu}$  is

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anti-symmetric tensor and Lorentz invariant and  $[x^\mu, x^\nu]_\star$  is commutator under the star product. NC field theories on flat spaces are given by the replacement of ordinary products with the Moyal-products and realized as deformed theories from the commutative ones. Moyal product for ordinary fields  $f(x)$  and  $g(x)$  is explicitly defined by

$$\begin{aligned} f(x) \star g(x) &= \exp\left(\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x'^\mu} \frac{\partial}{\partial x''^\nu}\right) f(x') g(x'')_{x=x'=x''} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu} \frac{\partial f}{\partial x'^\mu} \frac{\partial g}{\partial x''^\nu} + \mathcal{O}(\theta^2). \end{aligned}$$

this product obeys associative property  $f \star (g \star h) = (f \star g) \star h$ , if we apply the commutative limit  $\theta^{\mu\nu} \rightarrow 0$  then above expression will reduce to ordinary product as  $f \star g = f.g$ .

We are familiar that Lax equation is a nice representation of integrable systems, the form of Lax equation on deformed space is the same as it has on ordinary space, here ordinary product is replaced by the star product. The Lax equation involves two linear operators, these operators may be differential operators or matrices[7-12]. If  $A$  and  $B$  are the linear operators then Lax equation is given by  $A_t = [B, A]_\star$  where  $[B, A]_\star$  is commutator under the star product, this Lax pair formulasim is also helpful to construct the DT of integrable systems. Now consider a linear system  $\Psi_x = A(x, t)\Psi$  and  $\Psi_t = B(x, t)\Psi$ , the compatibility of this system yields  $A_t - B_x = [B, A]_\star$  which is called zero curvature condition[13-16], further let we express the commutator  $[\cdot]_-$  and anti-commutator  $[\cdot]_+$  without writing the  $\star$  as subscript then zero curvature condition may written as  $A_t - B_x = [B, A]_-$ .

In this paper, i have applied Lax pair formalism for the representation of NC P-II equation and this work also involves the explicit description of DT of this equation. Finally, i derive the multi-soliton solution of NC P-II equation in terms of quasideterminants.

## 2. Linear representation

Many integrable systems possess the linear representation on ordinary as well as on NC spaces, this representation is also known by the Lax representation, matrix P-II equation on ordinary space has this kind of representation [18]. Here, we will see that how the NC P-II equation arises from the compatibility condition of following linear systems

$$\Psi_\lambda = A(z; \lambda)\Psi, \Psi_z = B(z; \lambda)\Psi,$$

where  $A(z; \lambda)$  and  $B(z; \lambda)$  are matrices. The compatibility condition  $\Psi_{z\lambda} = \Psi_{\lambda z}$  implies

$$A_z - B_\lambda = [B, A]_- \quad (1)$$

above expression is similar to zero curvature condition and an alternative linear representation of NC P-II equation with matrices  $A$  and  $B$  defined as under

$$A = \begin{pmatrix} 8i\lambda^2 + iv^2 - 2iz & -iv_z + \frac{1}{4}C\lambda^{-1} - 4\lambda v \\ iv_z + \frac{1}{4}C\lambda^{-1} - 4\lambda v & -8i\lambda^2 - iv^2 + 2iz \end{pmatrix}$$

$$B = \begin{pmatrix} -2i\lambda & v \\ v & 2i\lambda \end{pmatrix}$$

where the  $\lambda$  is a commuting parameter and  $C$  is a constant. Now we can easily evaluate the following values

$$A_z - B_\lambda = \begin{pmatrix} iv_z v + ivv_z & -iv_{zz} - 4\lambda v_z \\ iv_{zz} - 4\lambda v_z & -iv_z v - ivv_z \end{pmatrix}$$

and

$$BA - AB = \begin{pmatrix} a & -b - 4\lambda v_z \\ b - 4\lambda v_z & -a \end{pmatrix}$$

where

$$a = iv_z v + ivv_z$$

$$b = 2iv^3 - 2i[z, v]_+ + iC$$

now by using the above values in (1) we have the following expression

$$\begin{pmatrix} 0 & -iv_{zz} + 2iv^3 - 2i[z, v]_+ + iC \\ iv_{zz} - 2iv^3 + 2i[z, v]_+ - iC & 0 \end{pmatrix} = 0$$

and finally we get

$$v_{zz} = 2v^3 - 2[z, v]_+ + C \quad (2)$$

equation (2) is similar to the NC P-II equation which is introduced by V. Retakh and V. Rubtsov [19] .

### 3. NC Symmetric Functions and Lax Representation

This section consists the Lax representation of the set of three equations of functions  $u_0$ ,  $u_1$  and  $u_1$  , this set is equivalent to the NC P-II equation [19]. Now consider the linear system

$$L_t \psi = \lambda \psi$$

the time evolution of  $\psi$  is given by

$$\psi_t = P \psi$$

and the above system is equivalent to the Lax equation

$$L_t = [P, L]_- \quad (3)$$

here  $\lambda$  is a spectral parameter and  $\lambda_t = 0$ . Now we take the Lax pair  $L, P$  in the following form

$$L = \left( \begin{array}{c|c|c} L_1 & O & O \\ \hline O & L_2 & O \\ \hline O & O & L_3 \end{array} \right)$$

and

$$P = \left( \begin{array}{c|c|c} P_1 & O & O \\ \hline O & P_2 & O \\ \hline O & O & P_3 \end{array} \right)$$

where

$$L_1 = \begin{pmatrix} 1 & 0 \\ -v_0 & -1 \end{pmatrix}, L_2 = \begin{pmatrix} 1 & 0 \\ -v_1 & -1 \end{pmatrix}, L_3 = \begin{pmatrix} -1 & 0 \\ -v_2 & 1 \end{pmatrix}$$

and the elements of matrix  $P$  are given by

$$P_1 = \begin{pmatrix} \rho_1 & 0 \\ 0 & -\rho_1 \end{pmatrix}, L_2 = \begin{pmatrix} -\rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, L_3 = \begin{pmatrix} -1 & 0 \\ \frac{1}{2}\sigma & 1 \end{pmatrix}$$

where

$$\rho_1 = -v_2 - \frac{1}{2}\alpha_0 v_0^{-1}, \rho_2 = -v_2 + \frac{1}{2}\alpha_1 v_1^{-1}, \sigma = v_0 - v_1 + 2v_2$$

and

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

When the above Lax matrices  $L$  and  $P$  are subjected to the Lax equation (3) we get

$$v_0' = v_2 v_0 + v_0 v_2 + \alpha_0$$

$$v_1' = -v_2 v_1 - v_1 v_2 + \alpha_1$$

$$v_2' = v_1 - v_0$$

the above system can be reduced to NC P-II equation (2) by eliminating  $v_0$  and  $v_1$ . For this Lax representation of symmetric functions, the quasideterminants  $|I + \mu L|_{ii}$  can not be expressed in terms of the expansion of symmetric functions [20].

#### 4. A Brief Introduction of Quasideterminants

This section is devoted to a brief review of quasideterminants introduced by Gelfand and Retakh [21]. Quasideterminants are the replacement for the determinant for matrices with noncommutative entries and these determinants plays very important role to construct the multi-soliton solutions of NC integrable systems [22,23], by applying the Darboux transformation. Quasideterminants are not just a noncommutative generalization of usual commutative determinants but rather related to inverse matrices, quasideterminants for the square matrices are defined as if  $A = a_{ij}$  be a  $n \times n$  matrix and  $B = b_{ij}$  be the inverse matrix of  $A$ . Here all matrix elements are supposed to belong to a NC ring with an associative product. Quasideterminants of  $A$  are defined formally as the inverse of the elements of  $B = A^{-1}$

$$|A|_{ij} = b_{ij}^{-1}$$

this expression under the limit  $\theta^{\mu\nu} \rightarrow 0$ , means entries of  $A$  are commuting, will reduce to

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}$$

where  $A^{ij}$  is the matrix obtained from  $A$  by eliminating the  $i$ -th row and the  $j$ -th column. We can write down more explicit form of quasideterminants. In order to see it, let us recall the following formula for a square matrix

$$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix} \quad (4)$$

where  $A$  and  $D$  are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a  $2 \times 2$  matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed  $2 \times 2$  matrix. So the explicit forms of quasideterminants are given iteratively by the following formula

$$|A|_{ij} = a_{ij} - \sum_{p \neq i, q \neq j} a_{iq} |A^{ij}|_{pq}^{-1} a_{pj}$$

the number of quasideterminant of a given matrix will be equal to the numbers of its elements for example a matrix of order 3 has nine quasideterminants. It is sometimes convenient to represent the quasi-determinant as follows

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{in} & \cdots & a_{ni} & \cdots & a_{nn} \end{vmatrix}.$$

Let us consider examples of matrices with order 2 and 3, for  $2 \times 2$  matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

now the quasideterminants of this matrix are given below

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{22}^{-1}a_{21}$$

$$|A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{22}a_{21}^{-1}a_{12}$$

$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{11}a_{12}^{-1}a_{22}$$

$$|A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}a_{11}^{-1}a_{12}.$$

The number of quasideterminant of a given matrix will be equal to the numbers of its elements for example a matrix of order 3 has nine quasideterminants. Now we consider the example of  $3 \times 3$  matrix, its first quasideterminants can be evaluated in the following way

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - a_{12}Ma_{21} - a_{13}Ma_{31} - a_{12}Ma_{31} - a_{13}Ma_{31}$$

where  $M = \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1}$ , similarly we can evaluate the other eight quasideterminants of this matrix.

## 5. Darboux transformation

Darboux transformations play a very important role to construct the multi-soliton solutions of integrable systems these transformations are deduced from the given linear system of integrable systems. To derive the DT of NC P-II equation we consider its linear systems with the column vector  $\psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ . Now the linear system will become

$$\begin{pmatrix} \chi \\ \Phi \end{pmatrix}_\lambda = \begin{pmatrix} 8i\lambda^2 + iv^2 - 2iz & -iv_z + \frac{1}{4}C\lambda^{-1} - 4\lambda v \\ iv_z + \frac{1}{4}C\lambda^{-1} - 4\lambda v & -8i\lambda^2 - iv^2 + 2iz \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} \chi \\ \Phi \end{pmatrix}_z = \begin{pmatrix} -2i\lambda & v \\ v & 2i\lambda \end{pmatrix} \begin{pmatrix} \chi \\ \Phi \end{pmatrix}. \quad (6)$$

The standard transformations [24,25,26] on  $\chi$  and  $\Phi$  are given below

$$\chi \rightarrow \chi[1] = \gamma\Phi - \gamma_1\Phi_1(\gamma_1)\chi_1^{-1}(\gamma_1)\chi \quad (7)$$

$$\Phi \rightarrow \Phi[1] = \gamma\chi - \gamma_1\chi_1(\gamma_1)\Phi_1^{-1}(\gamma_1)\Phi \quad (8)$$

where  $\chi$ ,  $\Phi$  are arbitrary solutions at  $\gamma$  and  $\chi_1(\gamma_1)$ ,  $\Phi_1(\gamma_1)$  are the particular solutions at  $\gamma = \gamma_1$  of equations (5) and (6), these equations will take the following forms under the transformations (7) and (8)

$$\begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix}_\lambda = \begin{pmatrix} 8i\lambda^2 + iv^2[1] - 2iz & -iv_z[1] + \frac{1}{4}C\lambda^{-1} - 4\lambda v[1] \\ iv_z[1] + \frac{1}{4}C\lambda^{-1} - 4\lambda v[1] & -8i\lambda^2 - iv^2[1] + 2iz \end{pmatrix} \begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix}_z = \begin{pmatrix} -2i\lambda & v[1] \\ v[1] & 2i\lambda \end{pmatrix} \begin{pmatrix} \chi[1] \\ \Phi[1] \end{pmatrix}. \quad (10)$$

Now from (6) and equation (10) we have the following expressions

$$\chi_z = -i\lambda\chi + v\Phi \quad (11)$$

$$\Phi_z = i\lambda\Phi + v\chi \quad (12)$$

and

$$\chi_z[1] = -i\lambda\chi[1] + v[1]\Phi[1] \quad (13)$$

$$\Phi_z[1] = i\lambda\Phi[1] + v[1]\chi[1]. \quad (14)$$

Now substituting the transformed values  $\chi[1]$  and  $\Phi[1]$  in equation (13) and then after using the (11) and (12) in resulting equation, we get

$$v[1] = \Phi_1\chi_1^{-1}v\Phi_1\chi_1^{-1}. \quad (15)$$

Equation (15) represents the Darboux transformation of NC P-II equation, where  $v[1]$  is a new solution of NC P-II equation, this shows that how the new solution is related to the seed solution  $v$ . By applying the DT iteratively we can construct the multi-soliton solution of NC P-II equation.

### 5.1. Quasideterminant solutions

The transformations (7) and (8) may be expressed in the form of quasideterminants, first consider the equation (7) in follows form

$$\chi[1] = \gamma_0\Phi_0 - \gamma_1\Phi_1(\gamma_1)\chi_1^{-1}(\gamma_1)\chi_0$$

or

$$\chi[1] = \left| \begin{array}{cc} \chi_1 & \chi_0 \\ \gamma_1\Phi_1 & \boxed{\gamma_0\Phi_0} \end{array} \right| = \delta_\chi^e[1] \quad (16)$$

similarly we can do for the equation (8)

$$\Phi[1] = \left| \begin{array}{cc} \Phi_1 & \Phi_0 \\ \gamma_1\chi_1 & \boxed{\gamma_0\chi_0} \end{array} \right| = \delta_\Phi^e[1] \quad (17)$$

we have taken  $\gamma = \gamma_0$ ,  $\chi = \chi_0$  and  $\Phi = \Phi_0$  in order to generalize the transformations in  $N$ th form. Further, we can represent the transformations  $\chi[2]$  and  $\Phi[2]$  by quasideterminants

$$\chi[2] = \left| \begin{array}{ccc} \chi_2 & \chi_1 & \chi_0 \\ \gamma_2\Phi_2 & \gamma_1\Phi_1 & \gamma_0\Phi_0 \\ \gamma_2^2\chi_2 & \gamma_1^2\chi_1 & \boxed{\gamma_0^2\chi_0} \end{array} \right| = \delta_\chi^o[2]$$

and

$$\Phi[2] = \left| \begin{array}{ccc} \Phi_2 & \Phi_1 & \Phi_0 \\ \gamma_2\chi_2 & \gamma_1\chi_1 & \gamma_0\chi_0 \\ \gamma_2^2\Phi_2 & \gamma_1^2\Phi_1 & \boxed{\gamma_0^2\Phi_0} \end{array} \right| = \delta_\Phi^o[2].$$



here superscripts  $e$  and  $o$  of  $\delta$  represent the even and odd order quasideterminants. The  $N$ th transformations for  $\delta_\chi^o[N]$  and  $\delta_\Phi^o[N]$  in terms of quasideterminants are given below

$$\delta_\chi^o[N] = \begin{vmatrix} \chi_N & \chi_{N-1} & \cdots & \chi_1 & \chi_0 \\ \gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \Phi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_N^{N-1} \Phi_N & \gamma_{N-1}^{N-1} \Phi_{N-1} & \cdots & \gamma_1^{N-1} \chi_1 & \gamma_0^{N-1} \Phi_0 \\ \gamma_N^N \chi_N & \gamma_{N-1}^N \chi_{N-1} & \cdots & \gamma_1^N \chi_1 & \boxed{\gamma_0^N \chi_0} \end{vmatrix}$$

and

$$\delta_\Phi^o[N] = \begin{vmatrix} \Phi_N & \Phi_{N-1} & \cdots & \Phi_1 & \Phi_0 \\ \gamma_N \chi_N & \gamma_{N-1} \chi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_N^{N-1} \chi_N & \gamma_{N-1}^{N-1} \chi_{N-1} & \cdots & \gamma_1^{N-1} \chi_1 & \gamma_0^{N-1} \chi_0 \\ \gamma_N^N \Phi_N & \gamma_{N-1}^N \Phi_{N-1} & \cdots & \gamma_1^N \Phi_1 & \boxed{\gamma_0^N \Phi_0} \end{vmatrix}$$

here  $N$  is to be taken as even. in the same way we can write  $N$ th quasideterminant representations of  $\delta_\chi^e[N]$  and  $\delta_\Phi^e[N]$ .

$$\delta_\chi^e[N] = \begin{vmatrix} \chi_N & \chi_{N-1} & \cdots & \chi_1 & \chi_0 \\ \gamma_N \Phi_N & \gamma_{N-1} \Phi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \Phi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_N^{N-1} \chi_N & \gamma_{N-1}^{N-1} \chi_{N-1} & \cdots & \gamma_1^{N-1} \chi_1 & \gamma_0^{N-1} \chi_0 \\ \gamma_N^N \Phi_N & \gamma_{N-1}^N \Phi_{N-1} & \cdots & \gamma_1^N \Phi_1 & \boxed{\gamma_0^N \Phi_0} \end{vmatrix}$$

and

$$\delta_\Phi^e[N] = \begin{vmatrix} \Phi_N & \Phi_{N-1} & \cdots & \Phi_1 & \Phi_0 \\ \gamma_N \chi_N & \gamma_{N-1} \chi_{N-1} & \cdots & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_N^{N-1} \Phi_N & \gamma_{N-1}^{N-1} \Phi_{N-1} & \cdots & \gamma_1^{N-1} \Phi_1 & \gamma_0^{N-1} \Phi_0 \\ \gamma_N^N \chi_N & \gamma_{N-1}^N \chi_{N-1} & \cdots & \gamma_1^N \chi_1 & \boxed{\gamma_0^N \chi_0} \end{vmatrix}.$$

Similarly, we can derive the expression for  $N$ th soliton solution from equation (15) by applying the Darboux transformation iteratively, now consider

$$v[1] = \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} v \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1}$$

where

$$\Lambda_1^\phi[1] = \Phi_1$$

$$\Lambda_1^\chi[1] = \chi_1$$

this is one fold Darboux transformation. The two fold Darboux transformation is given by

$$v[2] = \phi[1] \chi^{-1}[1] v[1] \phi[1] \chi^{-1}[1]. \quad (18)$$

We may rewrite the equation (16) and equation (17) in the following forms

$$\chi[1] = \begin{vmatrix} \chi_1 & \chi_0 \\ \gamma_1 \Phi_1 & \boxed{\gamma_0 \Phi_0} \end{vmatrix} = \Lambda_2^\chi[2]$$

$$\Phi[1] = \begin{vmatrix} \Phi_1 & \Phi_0 \\ \gamma_1 \chi_1 & \boxed{\gamma_0 \chi_0} \end{vmatrix} = \Lambda_2^\phi[2].$$

and equation (18) may be written as

$$v[2] = \Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1} \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} v \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} \Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1}$$

In the same way, we can derive the expression for three fold Darboux transformation

$$v[3] = \Lambda_3^\phi[3] \Lambda_3^\chi[3]^{-1} \Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1} \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} v \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} \Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1} \Lambda_3^\phi[3] \Lambda_3^\chi[3]^{-1}.$$

here

$$\chi[2] = \begin{vmatrix} \chi_2 & \chi_1 & \chi_0 \\ \gamma_2 \Phi_2 & \gamma_1 \Phi_1 & \gamma_0 \Phi_0 \\ \gamma_2^2 \chi_2 & \gamma_1^2 \chi_1 & \boxed{\gamma_0^2 \chi_0} \end{vmatrix} = \Lambda_3^\chi[3]$$

and

$$\Phi[2] = \begin{vmatrix} \Phi_2 & \Phi_1 & \Phi_0 \\ \gamma_2 \chi_2 & \gamma_1 \chi_1 & \gamma_0 \chi_0 \\ \gamma_2^2 \Phi_2 & \gamma_1^2 \Phi_1 & \boxed{\gamma_0^2 \Phi_0} \end{vmatrix} = \Lambda_3^\phi[3].$$

Finally, by applying the transformation iteratively we can construct the  $N$ -fold Darboux transformation

$$v[N] = \Lambda_N^\phi[N] \Lambda_N^\chi[N]^{-1} \Lambda_{N-1}^\phi[N-1] \Lambda_{N-1}^\chi[N-1]^{-1} \dots \Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1} \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1} v \Lambda_1^\phi[1] \Lambda_1^\chi[1]^{-1}$$

$$\Lambda_2^\phi[2] \Lambda_2^\chi[2]^{-1} \dots \Lambda_{N-1}^\phi[N-1] \Lambda_{N-1}^\chi[N-1]^{-1} \Lambda_N^\phi[N] \Lambda_N^\chi[N]^{-1}$$

by considering the following substitution

$$\Theta_N[N] = \Lambda_N^\phi[N] \Lambda_N^\chi[N]^{-1}$$

in above expression, we get

$$v[N] = \Theta_N[N] \Theta_{N-1}[N-1] \dots \Theta_2[2] \Theta_1[1] v \Theta_1[1] \Theta_2[2] \dots \Theta_{N-1}[N-1] \Theta_N[N]$$

or

$$v[N] = \Pi_{k=0}^{N-1} \Theta_{N-k}[N-k] V \Pi_{j=N-1}^0 \Theta_{N-j}[N-j]$$

here we present only the  $N$ th expression for odd order quasideterminants  $\Lambda_N^\phi[N]$  and  $\Lambda_N^\chi[N]$

$$\Lambda_{2N+1}^\phi[2N+1] = \begin{vmatrix} \Phi_{2N} & \Phi_{2N-1} & \cdots & \Phi_1 & \Phi_0 \\ \gamma_{2N}\chi_{2N} & \gamma_{2N-1}\chi_{2N-1} & \cdots & \gamma_1\chi_1 & \gamma_0\chi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_{2N}^{2N-1}\chi_{2N} & \gamma_{2N-1}^{2N-1}\chi_{2N-1} & \cdots & \gamma_1^{2N-1}\chi_1 & \gamma_0^{2N-1}\chi_0 \\ \gamma_{2N}^{2N}\Phi_{2N} & \gamma_{2N-1}^{2N}\Phi_{2N-1} & \cdots & \gamma_1^{2N}\Phi_1 & \boxed{\gamma_0^{2N}\Phi_0} \end{vmatrix}$$

and

$$\Lambda_{2N+1}^\chi[2N+1] = \begin{vmatrix} \chi_{2N} & \chi_{2N-1} & \cdots & \chi_1 & \chi_0 \\ \gamma_{2N}\Phi_{2N} & \gamma_{2N-1}\Phi_{2N-1} & \cdots & \gamma_1\chi_1 & \gamma_0\Phi_0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \gamma_{2N}^{2N-1}\Phi_{2N} & \gamma_{2N-1}^{2N-1}\Phi_{2N-1} & \cdots & \gamma_1^{2N-1}\chi_1 & \gamma_0^{2N-1}\Phi_0 \\ \gamma_{2N}^{2N}\chi_{2N} & \gamma_{2N-1}^{2N}\chi_{2N-1} & \cdots & \gamma_1^{2N}\chi_1 & \boxed{\gamma_0^{2N}\chi_0} \end{vmatrix}$$

## 6. Conclusion

In this paper, I focused on the linear representations of NC P-II equation, I have also constructed Darboux transformation of this equation. Finally I derived its multi-soliton solution in terms of quasideterminants. Further motivations are to explore the other aspects of NC P-II equation such that its connection to other integrable equations, to construct its hierarchy and to check its the Painlevé property.

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